DEPTH OF SOME SQUARE FREE MONOMIAL IDEALS

DORIN POPESCU AND ANDREI ZAROJANU

ABSTRACT. Let $I \supsetneq J$ be two square free monomial ideals of a polynomial algebra over a field generated in degree ≥ 1, resp. ≥ 2 . Almost always when I contains precisely one variable, the other generators having degrees ≥ 2, if the Stanley depth of I/J is ≤ 2 then the usual depth of I/J is ≤ 2 too, that is the Stanley Conjecture holds in these cases.
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INTRODUCTION

Let K be a field, $S = K[x_1, \dots, x_n]$ be the polynomial algebra in n variables over K and $I \supsetneq J$ two square free monomial ideals of S. We assume that I, J are generated by square free monomials of degrees $\geq d$, resp. $\geq d+1$ for some $d \in \mathbf{N}$. Then $\operatorname{depth}_S I/J \geq d$ (see [4, Proposition 3.1], [12, Lemma 1.1]). Upper bounds of $\operatorname{depth}_S I/J$ are given by numerical conditions in [11], [12, Theorem 2.2], [13, Theorem 1.3] and [15, Theorem 2.4]. An important tool in the proofs is the Koszul homology, except in the last quoted paper, where the results are stronger, but the proofs are extremely short relying completely on some results concerning the Hilbert depth, which proves there to be a very strong tool (see [2], [17] and [6]). These results are inspired by the so called the Stanley Conjecture, which we explain below.

Let $P_{I\setminus J}$ be the poset of all square free monomials of $I \setminus J$ (a finite set) with the order given by the divisibility. Let $\mathcal{P}$ be a partition of $P_{I\setminus J}$ in intervals $[u, v] = \{w \in P_{I\setminus J} : u|w, w|v\}$, let us say $P_{I\setminus J} = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} \mathcal{P} = \min_i \deg v_i$ and the so called *Stanley depth* of I/J given by $\operatorname{sdepth}_S I/J = \max_{\mathcal{P}} \operatorname{sdepth} \mathcal{P}$, where $\mathcal{P}$ runs in the set of all partitions of $P_{I\setminus J}$ (see [4], [16]). The Stanley depth is not easy to handle, see [4], [14], [7], [5] for some of its properties.

Stanley's Conjecture says that $\operatorname{sdepth}_S I/J \geq \operatorname{depth}_S I/J$. Thus the Stanley depth of I/J is a natural combinatorial upper bound of $\operatorname{depth}_S I/J$ and the above results give numerical conditions to imply upper bounds of $\operatorname{sdepth}_S I/J$. When $J = 0$ the Stanley Conjecture holds either when $n \leq 5$ by [9], or when I is an intersection of four monomial prime ideals by [8], [10], or when I is an intersection of three primary ideals by [18], or when I is an almost complete intersection by [3].

Let r be the number of the square free monomials of degree d of I and B (resp. C) be the set of the square free monomials of degrees $d+1$ (resp. $d+2$) of $I \setminus J$. Set

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$s = |B|$, $q = |C|$. If either $s > r + q$, or $r > q$, or $s < 2r$ then $\operatorname{sdepth}_S I/J \leq d + 1$ and if the Stanley Conjecture holds then any of these numerical conditions would imply $\operatorname{depth}_S I/J \leq d + 1$. In particular this was proved directly in [13] and [15].

Now suppose that I is generated by one variable and some square free monomials of degrees ≥ 2. It is the purpose of our paper to show that almost always if $\operatorname{sdepth}_S I/J \leq 2$ then $\operatorname{depth}_S I/J \leq 2$ (see our Theorem 1.10). It is known already that $\operatorname{sdepth}_S I/J \leq 1$ implies $\operatorname{depth}_S I/J \leq 1$ (see [12, Theorem 4.3]) and so our Theorem 1.10 could be seen as a new step (small but difficult) in the study of Stanley's Conjecture.

1. Stanley depth of some square free monomial ideals

Let $I \supsetneq J$ be two square free monomial ideals of S. We assume that I, J are generated by square free monomials of degrees $\geq d$, resp. $\geq d+1$ for some $d \in \mathbf{N}$. As above B (resp. C) denotes the set of the square free monomials of degrees $d+1$ (resp. $d+2$) of $I \setminus J$.

Lemma 1.1. *Suppose that $d = 1$, $I = (x_1, \ldots, x_r)$ for some $1 \leq r < n$ and $J \subset I$ be a square free monomial ideal generated in degree ≥ 2. Let B be the set of all square free monomials of degrees 2 from $I \setminus J$. Suppose that* $\operatorname{depth}_S I/(J + ((x_j) \cap B)) = 1$ *for some $r < j \leq n$. Then* $\operatorname{depth}_S I/J \leq 2$.

Proof. Since $I/(J + ((x_j) \cap B))$ has a square free, multigraded free resolution we see that only the components of square free degrees of

$$\operatorname{Tor}^S_{n-1}(K, I/(J + (x_j) \cap B))) \cong H_{n-1}(x; I/(J + (x_j) \cap B))$$

are nonzero. Thus we may find $z = \sum_{i=1}^r y_i x_i e_{[n]\setminus\{1\}} \in K_{n-1}(x; I/(J + (x_j) \cap B)$, $y_i \in K$ inducing a nonzero element in $H_{n-1}(x; I/(J + (x_j) \cap B)$. Here we denoted $e_\tau = \wedge_{j\in\tau} e_j$ for a subset $\tau \subset [n]$. Then we see that

$$z' = \sum_{i=1}^r y_i x_i e_{[n]\setminus\{i,j\}} \in K_{n-2}(x; I/J)$$

induces a nonzero element in $H_{n-2}(x; I/J)$. Thus $\operatorname{depth}_S I/J \leq 2$ (see [1, Theorem 1.6.17]). □

Example 1.2. Let $n = 4$, $r = 2$, $d = 1$, $I = (x_1, x_2)$, $J = (x_1x_2)$, $B = \{x_1x_3, x_1x_4, x_2x_3, x_2x_4\}$. Then $F = I/(J + (x_1) \cap B) \cong (x_1, x_2)/((x_1) \cap (x_2, x_3, x_4))$ has sdepth and depth $= 1$, but $\operatorname{depth}_S I/J = 3$. Thus the statement of the above lemma can be false if $j < r$. More precisely, $\operatorname{depth}_S F = 1$ because $z = x_1e_{234}$ induces a nonzero element in $H_3(x; F)$ but e_1 is not present in e_{234}.

Proposition 1.3. *Suppose that $I \subset S$ is generated by $\{x_1, \ldots, x_r\}$ for some $1 \leq r \leq n$ and some square free monomials of degrees ≥ 2, and $x_ix_tx_k \in J$ for all $i \in [r]$ and $r < t < k \leq n$. Then* $\operatorname{depth}_S I/J \leq 2$.

Proof. First suppose that $I = (x_1, ..., x_r)$. If there exists $j > r$ such that $\operatorname{depth}_S I/(J+ (x_j) \cap B) = 1$ then we may apply the above lemma. Thus we may suppose that

$\operatorname{depth}_S I/(J+(x_j)\cap B)\geq 2$ for all $j>r$. Assume that $\operatorname{depth}_S I/J>2$. By decreasing induction on $r<t\leq n$ we show that $\operatorname{depth}_S I/(J+(x_t,\dots,x_n))\cap B)\geq 2$. We assume that $t<n$ and $\operatorname{depth}_S I/(J+(x_{t+1},\dots,x_n))\cap B)\geq 2$, $\operatorname{depth}_S I/(J+(x_t,\dots,x_n))\cap B)=1$. Set $L=(J+(x_t)\cap B)\cap(J+(x_{t+1},\dots,x_n)\cap B)$. In the following exact sequence

$$0\to I/L\to I/(J+(x_t)\cap B)\oplus I/(J+(x_{t+1},\dots,x_n)\cap B)\to I/(J+(x_t,\dots,x_n)\cap B)\to 0$$

the last term has the depth 1 and the middle the depth ≥ 2. By the Depth Lemma we get $\operatorname{depth}_S I/L=2$.

Remains to show that $\operatorname{depth}_S I/J=\operatorname{depth}_S I/L$. Note that there exist no $c\in C$ multiple of x_tx_j for some $r<t<j\leq n$ by our hypothesis. Thus $L=J$. Then it follows $\operatorname{depth}_S I/J=2$ which contradicts our assumption. The induction ends for $t=r+1$ and we get $\operatorname{depth}_S I/(J+(x_{r+1},\dots,x_n)\cap B)=2$; but this is not possible (see for example [12, Lemma 1.8]).

Now suppose that $I=U+V$, where $U=(x_1,...,x_r)$ and V is generated by some square free monomials of degrees ≥ 2. In the following exact sequence

$$0\to U/(U\cap J)\to I/J\to I/(U+J)\to 0$$

the first term has depth ≤ 2 from above and the last term is isomorphic with $V/(V\cap(U+J))$ and has depth ≥ 2 by [12, Lemma 1.1]. So by the Depth Lemma it follows that $\operatorname{depth} I/J\leq 2$. $\square$

Example 1.4. Let $n=4$, $I=(x_1,x_2,x_3)$, $J=(x_1x_3)$. Clearly, $B_1=\emptyset$, $B=\{x_1x_2,x_1x_4,x_2x_3,x_2x_4,x_3x_4\}$ and $C=\{x_1x_2x_4,x_2x_3x_4\}$. We have $s=5$, $r=3$, $q=2$ and so $s=r+q$. Note that each $c\in C$ is a multiple of a monomial of the form x_ix_j for some $1\leq i<j\leq 3$ and so $\operatorname{depth}_S I/J\leq 2$ by the above proposition. On the other hand, it is easy to see that $z=x_1e_2\wedge e_3-x_2e_1\wedge e_3+x_3e_1\wedge e_2$ induces a nonzero element in $H_2(x;I/J)$ and so again $\operatorname{depth}_S I/J\leq 2$.

Lemma 1.5. *If a monomial u of degree k from $I\setminus J$ has all multiples of degrees $k+1$ in J then* $\operatorname{depth} I/J\leq k$.

Proof. Renumbering the variables x we may suppose that $u=x_1\cdots x_k$. Then we see that $u(x_{k+1},...,x_n)=0$ so $\operatorname{Ann}_S u=(x_{k+1},...,x_n)\in\operatorname{Ass}_S I/J$. Thus $\operatorname{depth} I/J\leq k$. $\square$

Lemma 1.6. *Suppose that $J\subset I$ are square free monomial ideals generated in degree $\geq d+1$, respectively $\geq d$ and let V be an ideal generated by e square free monomials of degrees $\geq d+2$, which are not in I. Then* $\operatorname{sdepth}_S(I+V)/J\leq d+1$ *(resp.* $\operatorname{depth}_S(I+V)/J\leq d+1$*) implies that* $\operatorname{sdepth}_S I/J\leq d+1$ *(resp.* $\operatorname{depth}_S I/J\leq d+1$*). For the depth the converse is also true.*

Proof. By induction on e, we may consider only the case $e=1$, that is $V=\{v\}$. In the following exact sequence

$$0\to I/J\to(I+V)/J\to(I+V)/(I+J)\to 0$$

the last term is isomorphic with $(v)/((v)\cap(I+J))$ and has depth and sdepth $\geq d+2$. Then the first term has sdepth $\leq d+1$ by [14, Lemma 2.2] and depth $\leq d+1$ by the Depth Lemma. $\square$

Lemma 1.7. *Suppose that $I \subset S$ is generated by $x_1, \dots, x_r$ and a nonempty set E of square free monomials of degrees 2 in the variables $x_{r+1}, \dots, x_n$, and* $\operatorname{sdepth}_S I/J = 2$. *Let $x_1x_t \in B$ for some t, $r < t \leq n$, $I' = (x_2, \dots, x_r) + (B \setminus \{x_1x_t\})$, $J' = J \cap I'$ and $\mathcal{P}$ a partition of I'/J' with sdepth 3. Assume that any square free monomial $u \in S$ of degree 2, which is not in I, satisfies $x_1u \in J$. Then*

(1) *For any $a \in (B \setminus (x_2, \dots, x_r, x_1x_t)) \cap (x_t)$ with $x_1a \notin J$ the interval $[a, x_1a]$ is in $\mathcal{P}$.*
(2) *If $c = x_tx_ix_j \notin J$, $r < i < j \leq n$, $i, j \neq t$ and $x_1x_tx_i, x_1x_tx_j \notin J$ then $b = c/x_t \in B$ and if moreover $x_1b \notin J$ then c is not present in an interval $[a, c]$, $a \in B$ of $\mathcal{P}$.*

Proof. Let $a = x_tx_\nu$ be a monomial of $B \setminus (x_2, \dots, x_r, x_1x_t))$ with satisfies $x_1a \notin J$. Suppose that the interval $[a, x_1a]$ is not in $\mathcal{P}$. Then there exists in $\mathcal{P}$ an interval $[a, c]$ with $c \in C$. Thus x_1x_ν is in B and so in $\mathcal{P}$ there exists an interval $[x_1x_\nu, c']$, $c' \in C$,. We replace the interval $[x_1x_\nu, c']$ by $[x_1, x_1a]$ to get a partition of I/J with sdepth ≥ 3. However, such partition of I/J is not possible because $\operatorname{sdepth}_S I/J = 2$. Thus the interval $[a, x_1a]$ is in $\mathcal{P}$.

Now, let c be as in (2). We will show that $b = c/x_t \in B$. Indeed, if $b \notin B$ then $b \notin (x_1, \dots, x_r)$ because otherwise $b \in J$, which is false. Thus c can enter only in an interval $[a, c]$ for let us say $a = x_tx_i$. But this interval is not in $\mathcal{P}$ because a belongs to the interval $[a, x_1a]$. Contradiction! Thus c does not appear in the intervals of $\mathcal{P}$. Replacing $[a, x_1a]$ with $[a, c]$ in $\mathcal{P}$ we get another partition of I'/J' with sdepth 3, where the interval $[a, x_1a]$ is not present, contradicting (1).

Moreover suppose that $x_1b \notin J$. By (1), c can appear only in the interval $[b, c]$ because we have already the intervals $[x_tx_i, x_1x_tx_i]$, $[x_tx_j, x_1x_tx_j]$ in $\mathcal{P}$. Then we cannot have an interval $[b, x_1b]$ in $\mathcal{P}$ and so x_1b could appear in the interval, let us say $[x_1x_i, x_1b]$. Certainly, it is possible that x_1b will not appear at all in an interval of $\mathcal{P}$, but we may modify $\mathcal{P}$ to get this. Replace in $\mathcal{P}$ the intervals $[x_1x_i, x_1b]$, $[b, c]$, $[x_tx_i, x_1x_tx_i]$ by the intervals $[b, x_1b]$, $[x_tx_i, c]$, $[x_1x_i, x_1x_tx_i]$ and we get another partition of I'/J' with sdepth 3 but without the interval $[x_tx_i, x_1x_tx_i]$, contradicting again (1). □

Lemma 1.8. *Suppose that $I \subset S$ is generated by x_1 and a nonempty set E of square free monomials of degrees 2 in $x_2, \dots, x_n$ and* $\operatorname{sdepth}_S I/J = 2$. *Assume that $x_1a \notin J$ for all $a \in E$ and any square free monomial $u \in S$ of degree 2, which is not in I, satisfies $x_1u \in J$. Then* $\operatorname{depth}_S I/J \leq 2$.

Proof. Let $1 < t \leq n$ be such that $x_1x_t \in B$. We may assume that $a_1, \dots, a_k$, are all monomials of $(E \cap (x_t)) \setminus \{x_1x_t\}$. Set $I_t = (B \setminus \{x_1x_t\})$ and $J_t = J \cap I_t$. In the exact sequence

$$0 \to I_t/J_t \to I/J \to I/J + I_t \to 0$$

the last term has depth ≥ 2 because it is isomorphic with $(x_1)/(x_1) \cap (J + I_t)$ and $x_1x_t \notin J + I_t$. If $\operatorname{sdepth}_S I_t/J_t \leq 2$ then we get $\operatorname{depth}_S I_t/J_t \leq 2$ by [12, Theorem 4.3]. Applying the Depth Lemma we get $\operatorname{depth}_S I/J \leq 2$.

Thus we may assume that $\operatorname{sdepth}_S I_t/J_t \geq 3$ for all $1 < t \leq n$ such that $x_1x_t \in B$. Let $\mathcal{P} = \mathcal{P}_t$ be a partition of I_t/J_t with sdepth $= 3$. By the above lemma the intervals $[a_j, x_1a_j]$, $1 \leq j \leq k$ are in $\mathcal{P}$.

Suppose that $c = x_ix_jx_t \in C$, $i, j, t > 1$ and $x_jx_t, x_ix_t \in E$. Then $a = x_ix_j \in E$ by the above lemma. By our hypothesis we have $x_1a, x_1x_jx_t, x_1x_ix_t \in C$. Thus c cannot appear in an interval of $\mathcal{P}$ using again the above lemma.

For $b = x_1x_i \in B$, $\mathcal{P}$ must contain some intervals of the form $[x_1x_i, x_1a'_i]$ for some $a'_i \in E$. Certainly $a'_i \notin (x_t)$ because we saw that all a_j, $1 \leq j \leq k$ enter already in the intervals $[a_j, x_1a_j]$. Then these a'_i enter in some intervals $[a'_i, c'_i]$ with $c'_i \in (C \setminus (x_1))$. If $c'_i \in (a_j)$ for some a_j, $1 \leq j \leq k$ then the third divisor of c'_i of degrees 2 is in B too, and as above c'_i cannot appear in an interval of $\mathcal{P}$. Contradiction! Thus $c'_i \in (C \setminus (x_1, a_1, \ldots, a_k))$.

Let $I' = (x_1x_t, a_1, \ldots, a_k)$, $J' = J \cap I'$. We have seen that $c'_i \notin I'$. In the exact sequence

$$0 \to I'/J' \to I/J \to I/J + I' \to 0$$

we show that the last term has sdepth ≥ 3. Let $a'_i = x_ix_{\nu_i} \in B$ for some $1 < \nu_i \leq n$. We may suppose that $t > 2$, $x_1x_2 \in B$ and we see that the intervals $[x_1, x_1a'_2]$, $[x_1x_i, x_1a'_i]$, $i > 2$, $i \neq \nu_2$, $[a'_i, c'_i]$ induce with the help of $\mathcal{P}$ a partition of $I/J + I'$ with sdepth 3. Indeed, the only possible problem is that in $\mathcal{P}$ could appear some intervals of type $[a, ax_t]$ for some $a \in (E \setminus (x_t))$, $c = ax_t$ being the least common multiple of two (a_j). But this is not possible as we saw above. By [14, Lemma 2.2] we get $\operatorname{sdepth}_S I'/J' \leq 2$ and so $\operatorname{depth}_S I'/J' \leq 2$ by [12, Theorem 4.3]. Applying the Depth Lemma we get as $\operatorname{depth}_S I/J \leq 2$.

Proposition 1.9. *Suppose that $I \subset S$ is generated by x_1 and a nonempty set E of square free monomials of degrees 2 in $x_2, \ldots, x_n$ and $\operatorname{sdepth}_S I/J = 2$. Let $E' = \{a \in E : x_1a \in C\}$ and $E'' = E \setminus E'$. Assume that any square free monomial $u \in S$ of degree 2, which is not in I, satisfies $x_1u \in J$ and one of the following conditions hold:*

(1) $|E''| \leq |C \setminus (x_1, E')|$

(2) $|E''| > |C \setminus (x_1, E')|$ *and* $|B| \neq |C| + 1$.

Then $\operatorname{depth}_S I/J \leq 2$.

Proof. If $E'' = \emptyset$ then we apply the above lemma. Apply induction on $|E''|$. If $E' = \emptyset$ then $C \cap (x_1) = \emptyset$ and the conclusion follows from Lemma 1.5. Let $E'' = \{a_1, \ldots, a_k\}$, $k > 0$. We claim that we may reduce our problem to the case when $(C \setminus (x_1)) \subset (E'')$. Indeed, otherwise let $c \in (C \setminus (x_1, E''))$. Then there exists $b \in E'$ such that $c \in (b)$. Choose t, $1 < t \leq n$ such that $x_t | b$. Then x_1x_t divides $x_1b \in C$ and so it is in B. Set $I' = (B \setminus \{x_1x_t\})$, $J' = J \cap I'$. In the following exact sequence

$$0 \to I'/J' \to I/J \to I/(I' + J) \to 0$$

the last term is isomorphic with $(x_1)/(x_1) \cap (I' + J)$ and has depth ≥ 2 because $x_1x_t \notin (I' + J)$. If $\operatorname{sdepth}_S I'/J' \leq 2$ then by [12, Theorem 4.3] we get $\operatorname{depth}_S I'/J' \leq 2$ and using the Depth Lemma it follows $\operatorname{depth}_S I/J \leq 2$.

Thus we may suppose that $\operatorname{sdepth}_S I'/J' \geq 3$ and let $\mathcal{P} = \mathcal{P}_t$ be a partition of I'/J' with sdepth 3. By Lemma 1.7 (see also the above lemma), $\mathcal{P}$ may contain

some disjoint intervals $[x_1x_i, x_1b_i']$, $[b_i', c_i']$, for some $b_i' \in E'$, $c_i' \in C \setminus (x_1)$, $i \neq 1, t$ with $x_1x_i \in B$, $[b', x_1b']$ for $b' \in E' \setminus \{\{b_i'\}\}$ and $[a_j, c_j]$, $j \in [k]$, $c_j \in C$. As in the proof of the above lemma we have $b_i' \notin (x_t)$. Thus the above b is not one of b_i' and enters in $\mathcal{P}$ in the interval $[b, x_1b]$. Note that c is not among $\{\{c_j\}\}$ because is not in (E''). On the other hand, if $c = c_i'$ then should be divisible by b and b_i', both being from E'. Then by Lemma 1.7 applied for a t' given by the only one common variable $x_{t'}$ of b, b_i', the third divisor $u = c/x_{t'}$ of degree 2 of c is in E, and $x_1u \in J$ because c can enter in an interval $[u, c]$ of a partition $\mathcal{P}_{t'}$. Thus $u \in E''$ and so $c \in (E'')$, which is false. Then we may replace the interval $[b, x_1b]$ by $[b, c]$, which is again false because all intervals $[b', x_1b']$, $b' \in (E') \cap (x_t)$ should be present in $\mathcal{P}$ by Lemma 1.7. This proves our claim. Also note that $|C \setminus (x_1)| \geq |B \cap (x_1)| - 1 + k$.

Then we may assume that $(C \setminus (x_1)) \subset (E'')$. We may suppose that $c_i \in (E')$ if and only if $p < i \leq k$ for some $0 \leq p \leq k$. Moreover, we will arrange to have as many as possible c_j outside (E'). If $c' \in (C \setminus (x_1))$ is a multiple of let say a_{p+1}, but $c' \notin (E')$. We may replace in the above intervals c_{p+1} by c', the effect being the increasing of p. Thus after such procedure we may suppose that either $p = k$, or there exist no c in $(C \setminus (x_1, c_1, \ldots, c_p)) \cap (a_{p+1}, \ldots, a_k)$ which is not in (E').

If $p = k$ then set $I'' = (x_1, E')$, $J'' = I'' \cap J$ and see that in the exact sequence

$$0 \to I''/J'' \to I/J \to I/(I'' + J) \to 0$$

the last term is isomorphic with $(E'')/(E'') \cap (I'' + J)$ and has sdepth 3 because the intervals $[a_j, c_j]$, $j \in [k]$ gives a partition with sdepth 3. Then $\operatorname{sdepth}_S I''/J'' \leq 2$ by [14, Proposition 2.2] and we get $\operatorname{depth}_S I''/J'' \leq 2$ by Lemma 1.8. Using the Depth Lemma it follows $\operatorname{depth}_S I/J \leq 2$.

Next suppose that $p < k$. Then $(C \setminus (x_1, c_1, \ldots, c_p)) \cap (a_{p+1}, \ldots, a_k) \subset (E')$. We may choose $c_1, \ldots, c_p$ from the beginning (it is possible to make such changes in $\mathcal{P}$) such that $e = |\{i : c_i \notin (a_{p+1}, \ldots, a_k)\}|$ is maxim possible and renumbering a_j, $j \leq p$ we may suppose that $c_i \notin (a_{p+1}, \ldots, a_k)$ if and only if $i \in [e]$ for some $0 \leq e \leq p$.

Suppose that there exists $c \in C \setminus (x_1, c_1, \ldots, c_p)$ such that $c \notin E'$. Then c is not in $(a_{p+1}, \ldots, a_k)$ and necessary $c \in (a_1, \ldots, a_p)$. Assume that $c \in (a_i)$ for some $i \in [p]$. If $i > e$ then $c_i \in (a_{p+1}, \ldots, a_k)$, let us say $c_i \in (a_j)$ for some $j > p$ and we may change c_j by c_i and replace c_i by c increasing p because $c_i \notin E'$. This is not possible since p was maxim given. Thus $i \leq e$ and so $e > 0$. If $c_i \in (a_{e+1}, \ldots, a_p)$, let us say $c_i \in (a_p)$ then we may replace c_p by c_i and c_i by c increasing e which is also not possible. Thus $c_i \notin (a_{e+1}, \ldots, a_p)$.

Then set $I_e = (x_1, B \setminus \{a_1, \ldots, a_e\})$, $J_e = I_e \cap J$. In the exact sequence

$$0 \to I_e/J_e \to I/J \to I/(I_e + J) \to 0$$

the last term has sdepth 3 because we may write there the intervals $[a_i, c_i], i \in [e]$ since $c_i \notin I_e$. By [14, Proposition 2.2] it follows that $\operatorname{sdepth}_S I_e/J_e \leq 2$ and so $\operatorname{depth}_S I_e/J_e \leq 2$ by induction hypothesis on $|E''|$. Using the Depth Lemma it follows $\operatorname{depth}_S I/J \leq 2$.

Now suppose that there exist no such c, that is $C \setminus (x_1, E') = \{c_1, \ldots, c_p\}$. Thus $p = |C \setminus (x_1, E')|$ and so we end the case when the condition (1) holds. Now suppose that the condition (2) holds, in particular $k > p$ and $s = |B| \neq 1 + q$ for $q = |C|$. If

$s > 1+q$ then we end with [13]. Suppose that $s < 1+q$. Then there exists a $c \in C$ which does not appear in an interval $[b, c]$ for some $b \in (B \setminus \{x_1x_t\})$. Note that c cannot be a c_j for $j \in [p]$ and so $c \in (E')$, let us say $c \in (a)$ for some $a \in E'$. Let j be such that $x_j|a$. We have $x_1x_j \in B$ and there exists as above a partition $\mathcal{P}_j$ with sdepth 3. Let $I_a = (B \setminus \{a\})$, $J_a = I_a \cap J$. We see that $\mathcal{P}_j$ induces a partition $\mathcal{P}_a$ of I_a/J_a with sdepth 3 replacing the interval $[a, x_1a]$ from $\mathcal{P}_j$ with $[x_1x_j, x_1a]$.

In $\mathcal{P}_a$ there is an interval $[x_1x_t, x_1a_1'']$ for some $a_1'' = x_tx_i \in E'$. We have $a_1'' \neq a'$ because otherwise we may change in $\mathcal{P}_t$ the interval $[a_1'', x_1a_1'']$ by $[a_1'', c]$, which is false. Then there is in $\mathcal{P}_a$ an interval $[a_1'', c_1'']$. If c_1'' is not a c_b as above then we may replace in $\mathcal{P}_t$ the interval $[a_1'', x_1a_1'']$ by $[a_1'', c_1'']$, which is again false. Thus $c_1'' = c_{b_1}$ for some $b_1 \in (B \setminus \{x_1x_t\})$. If $b_1 = a$ we may replace in $\mathcal{P}_t$ the intervals $[a_1'', x_1a_1'']$, $[b_1, c_1'']$ by $[a_1'', c_1'']$, $[b_1, c]$, which is false. Then there is in $\mathcal{P}_a$ an interval $[b_1, c_2'']$. By recurrence we find in $\mathcal{P}_a$ the intervals $[x_1x_t, x_1a_1'']$, $[a_1'', c_1'']$, $[a_2'', c_2''], \ldots$ which define a partition $\mathcal{P}_a$, where c is not present in an interval $[b, c]$, $b \in (B \setminus \{a\})$. Adding the interval $[a, c]$ to $\mathcal{P}_a$ we get a partition $\mathcal{P}'$ with sdepth 3 of I_B/J_B, where $I_B = (B)$, $J_B = I_B \cap J$. But then we replace in $\mathcal{P}'$ the intervals $[x_1x_t, x_1a_1'']$, $[x_1x_i, x_1a_1'']$ by $[x_1, x_1a_1'']$ and we get a partition of I/J with sdepth 3. Contradiction! $\square$

Theorem 1.10. *Suppose that $I \subset S$ is generated by x_1 and a nonempty set E of square free monomials of degrees 2 in $x_2, \ldots, x_n$ and* $\operatorname{sdepth}_S I/J = 2$. *Let $E' = \{a \in E : x_1a \in C\}$ and $E'' = E \setminus E'$. Assume that one of the following conditions holds:*

1. $|E''| \leq |C \setminus (x_1, E')|$
2. $|E''| > |C \setminus (x_1, E')|$ *and* $|B| \neq |C| + 1$.

Then $\operatorname{depth}_S I/J \leq 2$.

Proof. We may assume $n > 2$ and there exists $c = x_1x_{n-1}x_n \notin J$ after renumbering the variables x, otherwise we apply Proposition 1.3. Then $z = x_{n-1}x_n \notin J$.

First suppose that we may find c with $z \notin I$. Set $I' = (B \setminus \{x_1x_{n-1}, x_1x_n\})$ and $J' = I' \cap J$. Then necessary $B \supsetneq \{x_1x_{n-1}, x_1x_n\}$ and so $I' \neq J'$ because otherwise $\operatorname{sdepth}_S I/J = 3$. Note that no b dividing c belongs to I' and so $c \notin (J + I')$. In the following exact sequence

$$0 \to I'/J' \to I/J \to I/(I' + J) \to 0$$

the last term has sdepth ≥ 3 since $[x_1, c]$ is the whole poset of $(x_1)/(x_1) \cap (I' + J)$ except some monomials of degrees ≥ 3. It has also depth ≥ 3 because $x_{n-1}x_n \notin ((J + I') : x_1)$. The first term has sdepth $\leq \operatorname{sdepth}_S I/J = 2$ by [14, Lemma 2.2] and so it has depth ≤ 2 by [12, Theorem 4.3]. It follows $\operatorname{depth}_S I/J \leq 2$.

Next suppose that there exist no such c, that is any square free monomial $u \in S$ of degree 2, which is not in I satisfies $x_1u \in J$. We may assume that $C \subset (x_1, B)$ by Lemma 1.6. Now it is enough to apply Proposition 1.9. $\square$

Example 1.11. Let $n = 3$, $r = 1$, $I = (x_1, x_2x_3)$, $J = 0$. We have $c = x_1x_2x_3 \notin J$ and $x_2x_3 \in I$. Note also that $\operatorname{sdepth}_S I = \operatorname{depth}_S I = 2$.

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Dorin Popescu, "Simion Stoilow" Institute of Mathematics of Romanian Academy, Research unit 5, P.O.Box 1-764, Bucharest 014700, Romania

E-mail address: dorin.popescu@imar.ro

Andrei Zarojanu, Faculty of Mathematics and Computer Sciences, University of Bucharest, Str. Academiei 14, Bucharest, Romania

E-mail address: andrei_zarojanu@yahoo.com